10.1. Cut points and injectivity radius. Let M be a complete Riemannian manifold, and let $p \in M$.

(a) Show that if $\bar{p} \in M$ is a cut point of p along the unit speed geodesic $c\colon [0,l] \to M$, then p is a cut point of $\bar{p}$ along $\bar{c}\colon [0,l] \to M$, $\bar{c}(t) := c(l-t)$.

(b) Show that $\mathrm{inj}_p := d(p, \mathrm{Cut}(p))$ equals the supremum of all $r > 0$ such that $\exp_p|_{B_r}$ is injective, and also the supremum of all $r > 0$ such that $\exp_p|_{B_r} : B_r \to B(p,r)$ is a diffeomorphism.

Solution. (a) Note that $l = d(p,\bar{p}) = t_u$ is the cut value of $u := c'(0)$. By Lemma 6.4, (at least) one of the following holds: (1) $\bar{p}$ is conjugate to p along c, or (2) there exists a minimizing unit speed geodesic from p to $\bar{p}$ distinct from c. Let $\bar{u} := \bar{c}'(0)$. If (1) or (2) holds, then p is conjugate to $\bar{p}$ along $\bar{c}$, or there exists a minimizing unit speed geodesic from $\bar{p}$ to p distinct from $\bar{c}$, respectively. Then the last assertion of Lemma 6.4 shows that $t_{\bar{u}} \le l$. Since $\bar{c}$ is minimizing, $t_{\bar{u}} \ge l$, so equality holds.

(b) Use again Lemma 6.4 and Theorem 3.16. □

10.2. Non-negative Ricci curvature and maximal volume. Show that a complete Riemannian m-manifold M with $\mathrm{Ric} \ge 0$ and

$$\lim_{r\to\infty} \frac{\mathrm{vol}(B(p,r))}{V_{m,0}(r)} = 1$$

for some $p \in M$ is isometric to $\mathbb{R}^m$.

Solution. It follows from the assumptions and Theorem 6.7 (Bishop–Gromov) that $\mathrm{vol}(B(p,r)) = V_{m,0}(r)$ for all $r > 0$. The proof of Theorem 6.6 (Bishop) then shows that $B_r \cap U_p = B_r$ and $J\exp_p(v) = J_{m,0}(|v|) = 1$ for all $v \in B_r \subset TM_p$. Now Proposition 6.2 implies that all sectional curvatures at p are zero. If $q \in M$ is another point, then

$$1 \ge \frac{\mathrm{vol}(B(q,r))}{V_{m,0}(r)} \ge \frac{\mathrm{vol}(B(p, r - d(p,q)))}{V_{m,0}(r)} = \frac{V_{m,0}(r - d(p,q))}{V_{m,0}(r)} \to 1$$

for $r \to \infty$, and so the above argument applies to q as well. Hence, M is a Euclidean space form with $\mathrm{vol}(B(q,r)) = V_{m,0}(r)$ for all $q \in M$, and so must be isometric to $\mathbb{R}^m$. □

10.3. Growth of finitely generated groups.

(a) Verify that for a free group on $k \geq 2$ generators, with generating set $A = \{a_1, \dots, a_k\}$, the growth function satisfies

$$w_A(r) = \frac{k(2k-1)^r - 1}{k-1}$$

and hence $(2k-1)^r \leq w_A(r) \leq (2k+1)^r$ for all integers $r \geq 1$.

(b) Let $\tilde{M} = G$ be the nilpotent Lie group consisting of all 3×3 upper triangular real matrices with 1's on the diagonal (the *Heisenberg group*), and let Γ be the subgroup of all integer matrices. Then the quotient space $\tilde{M}/\Gamma$ is a compact 3-dimensional manifold with fundamental group Γ. Show that Γ is generated by $A = \{x, y\}$, where $x = I + e_{12}$ and $y = I + e_{23}$, and there exist constants $c_2 \geq c_1 > 0$ such that

$$c_1 r^4 \leq w_A(r) \leq c_2 r^4$$

for all integers $r \geq 0$. (Theorem 6.11 and Theorem 6.12 then show that $\tilde{M}/\Gamma$ does neither admit a metric with $\sec < 0$ nor a metric with $\mathrm{Ric} \geq 0$.) See the outline of Lemma 4 in J. Milnor, A note on curvature and fundamental group, J. Differential Geometry 2 (1968), 1–7.

Solution. (a) Since the group is free, every element γ is represented by a unique word of minimal length $|\gamma|_A$ in the alphabet $A \cup A^{-1}$, containing no redundant substring of the form aa^{-1} or $a^{-1}a$. For every integer $r \geq 1$, there exist exactly $2k(2k-1)^{r-1}$ such words of length r. Hence,

$$\begin{aligned} w_A(r) &= 1 + 2k + 2k(2k-1) + \ldots + 2k(2k-1)^{r-1} \\ &= 1 + 2k\frac{(2k-1)^r - 1}{(2k-1) - 1} = \frac{k(2k-1)^r - 1}{k-1}. \end{aligned}$$

Clearly $w_A(r) \geq (2k-1)^r$, and replacing all factors $(2k-1)$ in the above computation with $(2k+1)$ one sees that $w_A(r) \leq (2k+1)^r$.

(b) Writing $(i, j, k) \in \mathbb{Z}^3$ for the matrix $I + ie_{12} + je_{23} + ke_{13} \in \Gamma$, the group multiplication is

$$(i, j, k) \cdot (i', j', k') = (i + i', j + j', k + k' + ij').$$

Now if $x := (1, 0, 0)$, $y := (0, 1, 0)$, and $z := (0, 0, 1)$, then $z = xyx^{-1}y^{-1}$, and $(i, j, k) = z^k y^j x^i$. This shows that $A = \{x, y\}$ generates Γ. Furthermore, z is in the center of Γ, that is, z commutes with all $\gamma \in \Gamma$.

Since $zyx = xy$, every element $z^k y^s x^s$ with $0 \leq k \leq s^2$ can be expressed as a word of length $2s$ in x and y. It follows that if $1 \leq i \leq s$, $1 \leq j \leq s$, and $1 \leq k \leq s^2$, then

$z^k y^j x^i = y^{-(s-j)}(z^k y^s x^s)x^{-(s-i)}$ has word length at most $4s$. As there are s^4 distinct such elements $z^k y^j x^i$, this shows that $w_A(4s) \geq s^4$ for all integers $s \geq 1$, which implies a quartic lower bound for w_A.

Conversely, if $z^k y^j x^i$ has word length $\leq r$, one verifies that $|i|, |j| \leq r$ and $|k| \leq (r/2)^2$, which yields a quartic upper bound for $w_A(r)$.

$\square$